

### MATH 53H - Solutions to Problem Set I

1. We will use separation of variables. Note that we can write the equation as

$$x' = -(x - \frac{1}{2})^2 - (h - \frac{1}{4})$$

We distinguish the following three cases:

**a.**  $h < \frac{1}{4}$ . Let  $\alpha = \frac{1}{2} + \sqrt{\frac{1}{4} - h}$  and  $\beta = \frac{1}{2} - \sqrt{\frac{1}{4} - h}$ . Then we have

$$x' = -(x - \alpha)(x - \beta)$$

Clearly there exist two constant solutions  $x(t) = \alpha$  and  $x(t) = \beta$ . Hence suppose that  $x(t_0) \neq \alpha, \beta$  for some  $t_0$  and by continuity for all  $t$  close to  $t_0$ . Then we have that

$$\frac{x'}{x - \beta} - \frac{x'}{x - \alpha} = \alpha - \beta$$

Integrating this from  $t_0$  to  $t$  we get

$$\left| \frac{x - \beta}{x - \alpha} \right| = e^{(\alpha - \beta)(t - t_0)} \left| \frac{x(t_0) - \alpha}{x(t_0) - \beta} \right| = C' e^{2\sqrt{1/4 - h} \cdot t}$$

where  $C'$  is a non-zero constant. The quantity inside the absolute value is continuous and does not vanish, hence preserves sign, and we can remove the absolute value to obtain after some manipulation

$$x(t) = \frac{1}{2} + \sqrt{1/4 - h} \cdot \frac{C e^{2\sqrt{1/4 - h} \cdot t} + 1}{C e^{2\sqrt{1/4 - h} \cdot t} - 1}$$

where  $C = \pm C'$  a real non-zero constant.

**b.**  $h = \frac{1}{4}$ . We get  $x' = -(x - \frac{1}{2})^2$ . Arguing as in the previous case we get the solutions  $x(t) = \frac{1}{2}$  and

$$x(t) = \frac{1}{2} + \frac{1}{t - C}$$

for a constant  $C$ .

**c.**  $h > \frac{1}{4}$ . Then as in the above we get the solution

$$x(t) = \frac{1}{2} - \sqrt{h - 1/4} \cdot \tan(\sqrt{h - 1/4} \cdot t + C)$$

for a constant  $C$ .

**2.** Write  $x(t) = y(t) + z(t)$ , where  $x(t)$  is a solution of the given equation. Then we have  $y' + z' = ay + az + f(t)$  and since  $y(t)$  is a solution of the equation, we obtain  $z' = az$ . This implies that

$$(e^{-at}z)' = e^{-at}(z' - az) = 0 \Rightarrow e^{-at}z(t) = c \Rightarrow z(t) = ce^{at}$$

for a real constant  $c$ . Therefore we get  $x(t) = y(t) + ce^{at}$  and conversely we may check that this satisfies the equation for any  $c$ .

**3. (a)** Let  $b(t) = \int_0^t a(s)ds$  be an anti-derivative of  $a(t)$ . Then if  $x(t)$  is a solution of the equation we have

$$\begin{aligned}(e^{-b(t)}x)' &= e^{-b(t)}(x' - b'x) = e^{-b(t)}(x' - ax) = 0 \Rightarrow e^{-b(t)}x = c \\ &\Rightarrow x(t) = ce^{b(t)} = ce^{\int_0^t a(s)ds}\end{aligned}$$

for some real constant  $c$ .

**(b)** It is easy to check that the given solution satisfies the equation for any  $c \in \mathbb{R}$ .

**4. (a), (b)** An obvious solution is  $x(t) = 0$  for  $t \in \mathbb{R}$ . Suppose now that  $x(t_0) \neq 0$  for some  $t_0$ . Then  $x(t) \neq 0$  around  $t_0$  and therefore we obtain by integrating

$$\frac{x'}{x^2} = 1 \Rightarrow \frac{1}{x(t)} - \frac{1}{x(t_0)} = t - t_0 \Rightarrow x(t) = \frac{1}{t - C}$$

for a constant  $C \in \mathbb{R}$ . Depending on whether  $t_0$  is greater or less than  $C$ , the domain of definition is  $(C, +\infty)$  or  $(-\infty, C)$ .

**(c)** Consider the equation  $x' = \frac{\pi}{2}(1+x^2)$ . We can use separation of variables to find that the solution satisfying  $x(0) = 0$  is given by  $x(t) = \tan(\frac{\pi t}{2})$ , which is clearly defined only for  $-1 < t < 1$ .

**5.** By Exercise 3, we have that the general solution of the ODE is

$$x(t) = ce^{\int_0^t p(s)ds}$$

Hence all solutions are periodic with period  $T$  when

$$x(t+T) = x(t) \Leftrightarrow \int_0^{t+T} p(s)ds = \int_0^t p(s)ds \Leftrightarrow \int_t^{t+T} p(s)ds = 0$$

for all  $t$ . Let  $g(t) = \int_t^{t+T} p(s)ds$ . Then we have  $g'(t) = p(t+T) - p(t) = 0$  by the periodicity of  $p$  and thus  $g$  is constant. Therefore  $g(t) = g(0) = \int_0^T p(s)ds = 0$ , as we want.

**6. (i)** We have  $p_A(\lambda) = \det(\lambda I - A) = \lambda^2 + 1$ , hence the eigenvalues of  $A$  are  $\pm i$ . Therefore we know that the solutions are spanned by  $\Re(e^{it}) = \cos t$  and  $\Im(e^{it}) = \sin t$ . These are both  $2\pi$ -periodic and thus the same is true for all the solutions.

**(ii)** Letting  $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and solving explicitly we get  $a = 2d, b + c + 2d = 0$ .

Thus a solution is given by  $S = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ . Note that

$$(x \ y) S \begin{pmatrix} x \\ y \end{pmatrix} = (x - y)^2 + x^2$$

which is positive unless  $x = y = 0$ . Hence  $S$  is positive definite.

**(iii)** Let us denote  $f(t) = Q(x(t))$ . Then we have

$$\begin{aligned} f'(t) &= x'^T Sx + x^T Sx' = (Ax)^T Sx + x^T SAx = \\ &= x^T A^T Sx + x^T SAx = x^T (A^T S + SA)x = 0 \end{aligned}$$

by the definition of  $S$ . Thus  $f$  is constant, as we want.

**(iv)** Using part (iii), it suffices to show that the level sets of  $f$  are ellipses. The eigenvalues of  $S$  are  $\lambda_1 = \frac{3+\sqrt{5}}{2}, \lambda_2 = \frac{3-\sqrt{5}}{2}$  with corresponding eigenvectors  $v_1 = \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix}$ .

Note that the eigenvectors are orthogonal, hence if we normalize them and use them as columns of a matrix  $P$ ,  $P$  will be orthogonal, i.e.  $P^T = P^{-1}$ .

Moreover we will have  $S = P^T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P$  and thus

$$x^T Sx = (Px)^T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} (Px)$$

which implies that in the basis given by the eigenvectors the level sets of  $f$  are given by the equation of an ellipse of the form  $\lambda_1 u^2 + \lambda_2 v^2 = f(0) = Q(x(0))$  (in the new coordinates). Hence, since  $P$  is orthogonal and therefore an

isometry, the solution curve will be an ellipse in the original coordinates as well.

From this discussion it is clear that the principal axes of the ellipse are in the directions of  $v_1$  and  $v_2$  regardless of the initial condition. However their lengths are given by  $2\sqrt{\frac{Q(x(0))}{\lambda_1}}$  and  $2\sqrt{\frac{Q(x(0))}{\lambda_2}}$  and thus depend on the initial condition  $x(0)$ .